An Overview of Fourier Series

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Linear Algebra, redux

Recall that a vector space V is a collection of objects called "vectors" and an associated collection of "scalars," with operations for adding vectors and multiplying vectors by scalars. For our purposes, 'scalars' always refers to the set of real numbers $\mathbb{R}$.

A set $\{v_1, \ldots, v_n\}$ of vectors in a vector space is called a *basis* if for any vector v in the space, there's a unique list of scalars $c_1, \ldots c_n$ so that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

Such a sum is called a *linear combination* of the vectors v_i. As a side note, remember that the zero vector (which we merely write as 0) cannot appear in a basis.

The advantages of writing a vector as a linear combination depend on the situation. When solving ordinary differential equations, the vectors v_i might be fundamental solutions to an equation; the linear combinations are *all* solutions to the equation. Other times, we might know that an operator does nice things to the vectors v_i (for example, they could be eigenvectors), and would like to know what the operator does to other vectors.

There is one computational and theoretical problem. Generally there is no formula for finding the scalars c_i (short of solving a linear system of equations).

Inner Products

Now consider the typical dot product on Euclidean space. We normally write $u \cdot v$ for the dot product of the vectors u and v. Let's change notation (for a reason you'll see), and write

$$\langle u, v \rangle \qquad \text{instead of} \qquad u \cdot v.$$

Instead of "dot product," we'll refer to $\langle u, v \rangle$ as the *inner product*. There are five main properties that inner products have:

1. For any vectors u and v, the inner product $\langle u, v \rangle$ is a number.

2. For any vectors u and v, we have $\langle u, v \rangle = \langle v, u \rangle$.

3. For any vectors u and v and scalar a, we have $\langle au, v \rangle = a\langle u, v \rangle$.

4. For any vectors u, v and w, we have $\langle u + w, v \rangle = \langle u, v \rangle + \langle u, w \rangle$.

5. For any vector v, we have $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.

Now for the big definition.

Definition. *For any vector space, if an operation $\langle u, v \rangle$ satisfies all five properties listed above, we call it an* **inner product**. *We might then call the vector space an* **inner product space**.

Recall that two vectors in Euclidean space are orthogonal (or perpendicular) if and only if $\langle u, v \rangle = 0$. Also, the norm of a vector can be computed using $\|v\| = \sqrt{\langle v, v \rangle}$ for any vector v. These definitions work in any inner product space, so let's make another definition.

Definition. *We call two vectors u and v in an inner product space* **orthogonal** *if $\langle u, v \rangle = 0$. Given any vector v, define the* **norm** *as $\|v\| = \sqrt{\langle v, v \rangle}$.*

What do all of these notions do for us? Here's a guiding example.

Example 1. Find numbers c_1, c_2, c_3 so that

$$(1, 2, 0) = c_1(2, 1, 0) + c_2(-1, 2, 0) + c_3(0, 0, 1).$$

Notice that, among the vectors $(2, 1, 0), (-1, 2, 0), (0, 0, 1)$ the dot product of any pair is zero. Let's take advantage of this!

$$(1, 2, 0) \cdot (2, 1, 0) = c_1(2, 1, 0) \cdot (2, 1, 0) + c_2(-1, 2, 0) \cdot (2, 1, 0) + c_3(0, 0, 1) \cdot (2, 1, 0) = 5c_1 + 0 + 0.$$

On the other hand, $(1, 2, 0) \cdot (2, 1, 0) = 4$, so we find that $c_1 = 4/5$. Without solving a linear system, we deduced one of the coefficients with minimal effort. As an exercise for the reader, work out the other numbers using the same trick. □

The ideas in the example work in any inner product space and are the foundation what follows. One more definition first: the three basis vectors in the example form what is called an orthonormal set.

Definition. *If we have a set of vectors* $\{v_1, \ldots, v_n\}$ *so that the inner product of any two of them is zero, we call the set* **orthogonal**. *If furthermore* $\langle v_i, v_i \rangle = 1$ *for every vector in the set, we call the set* **orthonormal**.

The example can be generalized into a formula.

Theorem. *If* $\{v_1, \ldots, v_n\}$ *is both a basis and an orthogonal set in an inner product space, then for any vector* v,

$$v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 + \cdots + \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle} v_n.$$

Infinite Series and Bases

You've seen in linear algebra that vectors can be more interesting than pointy arrows living in three-dimensional space. Consider the collection of functions defined on the real numbers which have Taylor series expansions. This collection is a vector space, and the most natural basis is given by $1, x, x^2, x^3, x^4, \ldots$. After all, finding a Taylor series expansion is like writing a function *as a linear combination of powers of* x:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

From calculus we have formulas for the scalars, but they require a lot of work to compute (and in applications, the computations might be too numerically unstable). If only we could use the orthogonality trick to compute the scalars! But what inner product would we use?

Let's restrict ourselves to the interval $[-1, 1]$ for the moment. Given two smooth functions f and g we can define

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx.$$

You can check; this gives an inner product on the space of smooth functions (those defined on $[-1, 1]$ that is). There is another set of polynomials:

$$1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x), \frac{1}{8}(35x^4 - 30x^2 + 3), \ldots$$

which form an orthogonal basis of the space. These polynomials are called the Legendre polynomials, but they are not our concern here. We want to apply our methods to a new situation.

Fourier Series: Periods of 2π

In applications some of the most important functions are not polynomials, but periodic functions—those which satisfy an equation like

$$f(x + 2L) = f(x).$$

The number $2L$ is called a period of f. The simplest periodic functions (from the viewpoint of calculus) are sines and cosines. Often in practice we'd like to solve a differential equation with some sort of periodic input or boundary condition. But if our periodic input $f(x)$ isn't some sort of standard trigonometric function, what can we do? We'll try to write $f(x)$ as an infinite sum of sines and cosines.

Let's assume for now that $f(x)$ has period 2π. Suppose that we *could* write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)). \tag{1}$$

How could we find the coefficients? The answer is provided by our orthogonality trick. Define the inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx.$$

The following theorem is merely a calculus homework problem, but it makes everything else work.

Theorem. *Given integers m and n,*

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \begin{cases} 0 & \textit{if } m \neq n \\ 0 & \textit{if } m = n = 0 \\ \pi & \textit{if } m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = \begin{cases} 0 & \textit{if } m \neq n \\ 2\pi & \textit{if } m = n = 0 \\ \pi & \textit{if } m = n \end{cases}$$

In other words, $\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \ldots\}$ is an orthonormal set! Therefore, we get a formula for the coefficients in equation (1) by using our linear algebra theorem.

Theorem. *If equation (1) is true, then for positive integers n,*

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

The series in equation (1) is called the **Fourier series** of f. The general theory Fourier series is very difficult, although the results are easy to use. For instance, when does a function have a Fourier series expansion? The following are two very important facts to keep in mind.

1. If f is piecewise continuous and has left-hand and right-hand derivatives everywhere, then the Fourier series in equation (1) converges for every x. In particular, if f is piecewise smooth then the Fourier series converges everywhere.

2. Even assuming f is piecewise smooth, **the series need not converge to the value of f**. Instead, the series converges to

$$\frac{f(x-0) + f(x+0)}{2},$$

where we've defined the new notation

$$f(x-0) = \lim_{h \to 0+} f(x-h) \qquad \text{and} \qquad f(x+0) = \lim_{h \to 0+} f(x+h).$$

If f is continuous at the point x, then the series **does** converge to $f(x)$.

Fourier Series: General Periods

What if f is piecewise smooth and periodic, but its period $2L$ isn't 2π? Then $f(xL/\pi)$ is piecewise smooth and has period 2π. Playing with change of variables in the integrals gives us a slight generalization of what we had before.

Theorem. *Suppose that f is piecewise smooth and periodic with period $2L$. Then define*

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$$

The Fourier series

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi x}{L} \right) + b_n \sin \left(\frac{n\pi x}{L} \right) \right)$$

converges for every x to the value

$$\frac{f(x-0) + f(x+0)}{2},$$

which is just $f(x)$ if f is continuous at x.

There's one more generalization of interest. What if f is not periodic but only defined on an interval $[a, b]$? Then simply extend f to be periodic on the entire real line. For example, given $f(x) = x^2$ for $-\pi < x \le \pi$, extend f by defining $f(x) = (x - 2\pi)^2$ for $\pi < x \le 3\pi$ and $f(x) = (x - 4\pi)^2$ for $3\pi < x \le 5\pi$, etc. Just draw a picture.

Parseval's Identity

There's one last thing we'll mention about Fourier series. The proof of this result is too difficult for this class, but we'll use it to evaluate some interesting things later.

Theorem. *Given any function f on the interval $[-\pi, \pi]$ for which $\int_{-\pi}^{\pi} f(x)^2\,dx$ exists as a finite number,*

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\,dx.$$

Notice that we don't even need piecewise smooth as an assumption.